Fixed points & Functional iteration

Newton's method: $x_{n+1} = \underbrace{x_n - \frac{f(x_n)}{f'(x_n)}}_{\text{call this } F(x_n)}$

So, Newton's method is a <u>functional iteration</u>

$$x_{n+1} = F(x_n)$$

Suppose (in general now) that $\lim_{n\to\infty} x_n$ exists & $\lim_{n\to\infty} x_n = s$

Suppose also that F is continuous

Then $\lim\limits_{n\to\infty} F(x_n) = F\left(\lim\limits_{n\to\infty} x_n\right) = F(s)$

$\| \big($ here $x_{n+1} = F(x_n)\big)$

$\lim\limits_{n\to\infty} x_{n+1} = s$

So $s = F(s)$!!

Such a point is called a fixed pt. of F.

Very important & interesting problems (e.g. in optimization, algorithm design, diff. equations ....) can be reduced to finding the fixed pts of a functions F.

<u>Simple (but important case)</u>

$F: \quad C \longrightarrow C$ where C is a closed subset of $\mathbb{R}$.

Contractive mapping:

$$|F(x)-F(y)| \leq \lambda |x-y|$$

for some $\lambda < 1$

(can you see why it's called contractive?)

Theorem: Let $C \subseteq \mathbb{R}$ be closed. If $F: C \to C$ is contractive, then F has a unique fixed pt s. Moreover

$$s = \lim_{n\to\infty} x_{n+1} \text{ where } x_{n+1} = F(x_n)$$

for any starting pt $x_0 \in C$.

Proof: want to show that $(x_n)_{n=0}^{\infty}$ converges

but $x_n = x_0 + (x_1 - x_0) + \cdots + (x_n - x_{n-1})$

$$= \underbrace{\left(\sum_{i=1}^{n} (x_i - x_{i-1})\right)}_{\text{want this sequence to converge as } n\to\infty} + x_0$$

It suffices to show that $\sum_{i=1}^{n} |x_i - x_{i-1}|$ converges

but

$$|x_{i+1} - x_i| = |F(x_i) - F(x_{i-1})|$$

$$\xrightarrow{\text{contractive map}} \leqslant \lambda |x_i - x_{i-1}|$$

$$\vdots$$

$$\leqslant \lambda^i |x_1 - x_0|$$

So $\sum_{i=1}^{\infty} |x_i - x_{i-1}| \leqslant \sum_{i=1}^{\infty} \lambda^i |x_1 - x_0| = \frac{1}{1-\lambda} |x_1 - x_0|$

$\Rightarrow$ the sequence converges, so let $s = \lim_{n \to \infty} x_n$ and note that $s = F(s)$.

To see that s is unique, suppose t is a different fixed pt. so $t = F(t)$

$$|t - s| = |F(t) - F(s)| \leqslant \lambda |t - s|$$

but $\lambda < 1$ so $t = s$. $\blacksquare$

Exercise: Let $F(x) = a + b\sin(x)$ for some $a \in \mathbb{R}$ & $b \in \mathbb{R}$. For what values of a & b is F contractive? In that case, write a iteration to find the fixed pt of F.

Order of Convergence: Suppose $\begin{cases} x_{n+1} = F(x_n) \\ F(s) = s \end{cases}$

Let $e_n = x_n - s$

(then $\lim_{n\to\infty} e_n = 0$) & the order of convergence is the smallest integer k s.t. $F^{(k)}(s) \neq 0$.